Ostrowski-Reich Theorems

International Workshop on Numerical linear Algebra with Applications

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Thank to Professor Raymond Chan, Professor Bob Plemmons, and my all collaborators

for their nice invitation, supports, helps and collaborations!

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Iterative Methods:

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b.$$

Objective: x_k converges a solution

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Let $A = D - A_L - A_U$. $M = D$ or $M = D - \omega A_L$. Then, we study conditions for $\rho(M^{-1}N) < 1$.

Existence of Convergent Methods

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J. Yuan, Iterative Refinement Methods Using Splitting Methods. LAA, 273(1997) 199-214.

Ostrowski-Reich Theorem

Reich Theorem

Let A be symmetric with positive diagonal elements. Suppose that M is lower triangular part of A, and $N = A - M$. Then, all eigenvalues of $M^{-1}N$ within the unit circle if and only if A is positive definite.

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E. Reich, On the convergence of the classical iterative method for solving linear simultaneous equations, Ann. Math. Statist., 20(1949), 448-451.

Ostrowski Theorem

Let $A = D - E - E^*$ be symmetric with positive diagonal elements where E is strictly lower triangular of A. Suppose that

$$M = \frac{1}{\omega}(D - \omega E),$$

and $N = A - M$. Then, all eigenvalues of $M^{-1}N$ within the unit circle for $0 < \omega < 2$ if and only if A is positive definite.

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A.M. Ostrowski, On the linear iteration procedures for symmetric matrices, Rend. Mat. e Appl. (5)14 (1954) 140-163.

Householder-John Theorem

If A is hermitian and if $M^* + N$ is positive definite, then $\rho(M^{-1}N) < 1$ if and only if A is positive definite.

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A.S. Householder, On the convergence of matrix iterations, Oak Ridge Notional Laboratory Technical Report No. 1883, 1955.

F. John, Advanced Numerical Analysis, Lecture Notes, Department of Mathematics, New York University,1956.

Ortega-Plemmons Theorems

If A and $M^*A^{-*}A + N$ satisfy the condition

$$x^*Ax \neq 0, \quad x^*(M^*A^{-*}A + N)x > o \qquad (*)$$

for every x in some eigenset E of $M^{-1}N$, then $\rho(M^{-1}N) < 1$. Conversely, if $\rho(M^{-1}N) < 1$, then for each eigenvector x of H either (*) holds or else $x^*Ax = x^*(M^*A^{-*}A + N)x = 0$.

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Assume that$M^*A^{-*}A + N$ is positive definite. Then, $\rho(M^{-1}N) < 1$ if and only if A is positive definite.

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J.M. Ortega, and R.J. Plemmons, Extensions of the Ostrowski-Reich theorem for SOR iterations, LAA, 28(1971)177-191.

Let $A = M - N$ and $H = M^{-1}N$. Then,

$$A^*A - H^*A^*AH = (I - H)^*(M^*A + A^*N)(I - H)$$

$$= (I - H)^*(M^*M - N^*N)(I - H).$$

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$$E(H) = \{x \in C^n : \quad Hx = \lambda x, \quad x \neq 0\}.$$

Suppose that A is nonsingular. Then, $\rho(M^{-1}N) < 1$ if and only if $M^*A + A^*N$ is $E(M^{-1}N)-$positive definite.

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$E(H)-$positive definite means that for all $x \in E(H)$, $x^*Hx > 0$ where x is eigenvector of H.

Given A and B bounded linear operators on Hilbert space with $A+B$ positive definite, then, $A-B^*$ positive definite if and only if $\rho(A^{-1}B)<1$.

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M.P. Hanna, Generalized overrelexation and Gauss-Seidel convergence on Hibert Space, Proceedings of the American Math. Society, 35(1972)524-530.

Singular Case:

Keller Theorem

Let the singular matrix A with $A = M - N$ be hermitian where M is nonsingular. Assume that $M : R(A) = R(A^*) \to R(A)$ and $M^* + N$ is positive definite. Then, $M^{-1}N$ is semi-convergent if and only if A is positive semi-definite.

K.B. Keller, On the solution of singular and semi-definite linear systems by iteration, SIAM J. Numer. Anal., 2(1965) 281-290.

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Let A be a singular matrix with $A = M - N$ and $H = M^{-1}N$. Assume that $M : R(A^*) \to R(A)$. Then, if $M^*A + A^*N$ is positive definite on $E'(H)$, H is semi-convergent. Conversely, if H is semi-convergent, then $M^*A + A^*N$ is positive definite on $E'(H)$ or $x^*A^*Ax = x^*(M^*A + A^*N)x = 0$ for all $x \in N(A)$, where

$$E'(H) = E(H) \bigcap R(A^*) = \{x \in C^n, x \neq 0, \;\; \exists \lambda \in C, \;\; Hx = \lambda x, \;\; \lambda \neq 1\}.$$

Assume that a singular matrix A has the property $R(A) = R(A^*)$ and $M : R(A) \to R(A)$. If $N + M^*(A^\dagger)^*A \neq 0$ satisfy the condition

$$x^*Ax \neq 0, \quad \frac{x^*(M^*(A^\dagger)^*A + N)x}{x^*Ax} > 0 \qquad (**)$$

$\forall x \in E(H) \bigcap R(A)$, then H is semi-convergent. Conversely if H is semi-convergent, then (**) holds or

$$x^*AX = x^*(M^*(A^\dagger)^*A + N)x = 0.$$

Let A be a singular matrix with $index(A) \leq 1$. Assume that $x^*Mx \neq 0$ for all $x \in E(H)$. Then, H is semi-convergent if and only if

$$[Re(x^*(M+N)x][Re(x^*Ax)] \geq -[Im(x^*(M+N)x)][Im(x^*Ax)]$$

for all $x \in E(H)$.

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for all $x \in E(H)$.

Assume that $M^*(A^\dagger)^*A + N$ is positive definite and $Index(A) \leq 1$. Then, H is semi-convergent if and only if A is positive semi-definite.

There are more extensions on this theorem at different situations, like the SOR method, Regular splittings, AOR method, and TOR method, also generalized SOR, AOR mehods etc.

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Constrained Problem

$$\min_{x \in S} f(x)$$

$$\text{s.t. } g(x) \leq 0$$

where $f(x)$ is career suffering function, S is the set of experienced researchers in Optimization and Numerical Linear Algebra area, and $g(x)$ is difficulty constrained functions or other types of constrained functions.

Difficulties

Fresh Professor $\Rightarrow$ Junior Researcher

- Topics Choice
- Evaluation
- Funding Hunting
- Facilities
- Political Conditions
- Academic Atmosphere

$$\min_{x \in S} f(x)$$

$$\text{s.t. } g(x) \leq 0$$

where $f(x)$ is career suffering function, $S = \{$ Cesar Camacho, Raymond Chan, Gene Golub, Clovis Gonzaga, Apostolo, Hadjidimos, Carlos Humes, Alfredo Iusem, Nelson Maculan, José Mario Martínez, Jacob Palis, Alvaro de Pierro, Robert Plemmons, Robert Russel, $\ldots\}$

Jackson $\neq$ Jackson

I was going to Jackson at WY, but my ticket was to go to Jackson at NC in 1995.

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Jackson at WY is not Jackson at NC.

"Illustrate" $\neq$ "Show" or "Display"

Numerical experiments or results cannot show anything, but illustrate somethings.

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Numerical experiments or results cannot show anything, but illustrate somethings.

Bob always teaches me English and to improve my writing skills.

Thank someone is not bad Idea

Even someone rejected your paper, thank him is not bad idea because he made you improve your paper and think deeper.

Thank someone is not bad Idea

Even someone rejected your paper, thank him is not bad idea because he made you improve your paper and think deeper.

He teaches me to encourage all people and keep good relationship with all people as possible.

You are not the first one

One of my paper was rejected by chief-editor after almost 3 years even two referees recommended to accept the paper.

You are not the first one

One of my paper was rejected by chief-editor after almost 3 years even two referees recommended to accept the paper.

Bob teaches me to face difficult situation with good attitude.

Now I define my constrained functions:

- g_1 is minimal distance function with the center in Curitiba
- g_2 is the spirit support in all time.
- g_3 is co-authorship.
- g_4 is the speed of email answer.
- g_5 is the kindest hospitality for my visiting.
- g_6 is to help me to improve my English and paper quality.
- g_7 whenever possible to visit me in Brazil.

Solution is

Robert Plemmons

Bob gives me support and help all time on my career in all aspects.

Happy Birthday, Bob !!!

Conference News in Brazil

X Brazilian Workshop on Continuous Optimization
Celebrating Clovis Gonzaga's 70th birthday
Florianópolis, Santa Catarina, Brazil , March 17-21, 2014
http://www.impa.br/opencms/pt/eventos/store/evento1404

ICM will be at Rio de Janeiro in 2018. I was asked to organize one satellite conference at Foz do Iguaçu.